Properties of autowaves including transitions between the traveling and static solitary states

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An analytical investigation of shape, stability, and evolution of traveling and static solitary states—autosolitons (AS)—for the Rinzel-Keller-Koga-Kuramoto (RKKK) model of active media was carried out. The dependencies of the velocity and width of a traveling autosoliton on the bifurcation parameter at different ratios between characteristic lengths and times of activator and inhibitor were analyzed. It was shown that the traveling autosoliton loses stability at a velocity greater than those at which the corresponding solution disappears. It was found that the ASs velocity may become zero at some value of the bifurcation parameter, i.e., the traveling autosoliton transforms into a static one. At the same bifurcation point the static autosoliton loses stability with respect to the growth of fluctuations, which leads to the formation of a traveling or pulsating autosoliton. Transitions between traveling and static autosolitons are accompanied by hysteresis. It was also shown that besides this bifurcation point at certain parameters a "tricritical point" exists where the bifurcation of solutions in the form of traveling, static, and pulsating autosolitons is realized.

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I. INTRODUCTION

Interest in nonlinear phenomena in various nonequilibrium systems is growing. One of such striking phenomena is the formation of different dissipative structures [1] and autowaves [2], including static, pulsating, and traveling solitary eigenstates, so called autosolitons (ASs) [3–5].

It can be presumed from the general theory of ASs [3,4] that there may exist nonequilibrium systems where static, pulsating, and traveling ASs can be excited simultaneously. One of the goals of the present paper is to prove this presumption. For this purpose we will analyze the classical Rinzel-Keller-Koga-Kuramoto (RKKK) model of active media with diffusion, which has an exact analytical solution.

The RKKK model is described by the following equations:

$$\tau_\theta \frac{\partial \theta}{\partial t} = l^2 \frac{\partial^2 \theta}{\partial x^2} - [\theta + \eta - H(\theta - A)] , \quad (1)$$

$$\tau_\eta \frac{\partial \eta}{\partial t} = L^2 \frac{\partial^2 \eta}{\partial x^2} - \eta + \theta , \quad (2)$$

where

$$H(\theta - A) = \begin{cases} 1, & \theta \geq A, \\ 0, & \theta < A . \end{cases} \quad (3)$$

θ and η are state variables (temperature, density, etc); θ and η being activator and inhibitor, respectively [3,4].

For $L \ll l$ (more correctly, $L = 0$) Eqs. (1)–(3) are the well known Kinzel-Keller model admitting solutions in the form of traveling ASs [6,7]. Koga and Kuramoto investigated Eqs. (1)–(3) in another limiting case, $L \gg l$ [8]. They showed that for $L \gg l$ Eqs. (1)–(3) admit solutions in the form of static or pulsating ASs depending on the ratio of the parameters τ_θ and τ_η. The evolution of such ASs and other more complex dissipative structures is considered in [8–12] (see also [3–5]).

As follows from the general theory of ASs [3,4] the dissipative structures and autowaves (in particular, static, pulsating, and traveling ASs) may in principle coexist when the system's parameters lie within some narrow regions, in particular, when $\alpha \equiv \tau_\theta / \tau_\eta \ll 1$ and $\varepsilon \equiv l/L \ll 1$. To find these regions for the RKKK model we will analyze different types of solitary states and their stability for arbitrary ratios L/l and τ_θ/τ_η. Most attention will be concentrated on the traveling ASs because static AS behavior was investigated in [8–11].

II. SHAPE OF TRAVELING AUTOSOLITONS

To analyze the shape of an AS traveling with constant velocity V let us consider automodel solutions of Eqs. (1) and (2) in the form $\theta(x - Vt)$, $\eta(x - Vt)$. Substituting them into Eqs. (1) and (2) and using the dimensionless variables

$$\xi = \frac{x}{l} - v\frac{t}{\tau_\theta} \text{ and } v = \frac{V\tau_\theta}{l} , \quad (4)$$

we can rewrite these equations in the convenient form

$$-v\frac{d\theta}{d\xi} = \frac{d^2\theta}{d\xi^2} - [\theta + \eta - H(\theta - A)] , \quad (5)$$

$$-\alpha^{-1} v \frac{d\eta}{d\xi} = \varepsilon^{-2} \frac{d^2\eta}{d\xi^2} + \theta - \eta . \quad (6)$$

A solution describing a traveling AS in infinite media must satisfy the following boundary conditions:

$$\theta|_{\xi=\pm\infty} = \eta|_{\xi=\pm\infty} = 0 , \quad (7)$$

since the system becomes homogeneous ($\eta = \eta_h = 0$,

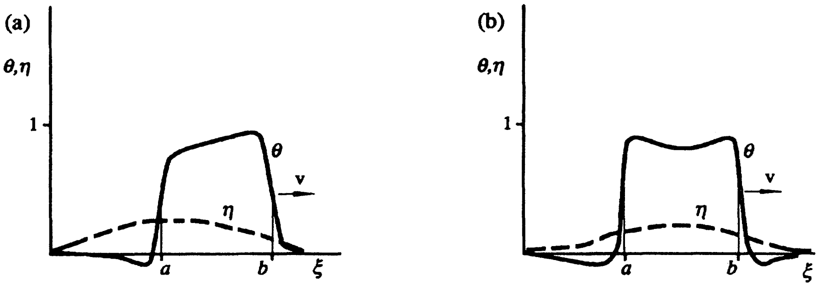

FIG. 1. Qualitative shape of traveling autosolitons, corresponding to Eqs. (A3) and (A4), for two limiting cases: (a) large drift length of the inhibitor ($\varepsilon^2 \gg \alpha$, $\alpha \ll 1$); (b) large diffusion length of the inhibitor ($\varepsilon \ll \alpha \ll 1$).

$\theta=\theta_h=0$) far from the center of the AS. Note also that at $v=0$ and $\varepsilon \ll 1$ the problem (5)–(7) describes the shape of static ASs [8,9].

A detailed analytical investigation of problem (5)–(7) is carried out in Appendix A. According to Eqs. (A3) and (A4), the shape of a traveling AS is determined by the ratio between the drift ($\tilde{L}=V\tau_\eta=v\alpha^{-1}l$) and diffusion ($L$) lengths of the inhibitor. When $\tilde{L} \gg L$ ($\alpha \ll \varepsilon v$) the shape of the traveling AS is essentially asymmetric and differs qualitatively from the shape of a static AS. In this case the minimum velocity of the AS $v_{\min} \simeq \sqrt{\alpha} = \sqrt{\tau_\theta/\tau_\eta} \ll 1$ (see Sec. III). Therefore the condition $\tilde{L}=V\tau_\eta \gg L$ is obviously satisfied when $\alpha \ll \varepsilon^2 \ll 1$. As follows from Eqs. (A3) and (A4), in this case the distribution of activator $\theta(\xi)$ varies nonmonotonically in the refractive region [at $\xi<a$ in Fig. 1(a)] and monotonically ahead of the ASs front wall [at $\xi>b$ in Fig. 1(a)].

When $\tilde{L}<L$ the shape of the traveling AS [Fig. 1(b)] is similar to that of the static AS. As the maximum velocity of the AS $v_{\max} \le 2/\sqrt{3}$ [see below Eq. (12)], the condition $\tilde{L}<L$ is valid when $\varepsilon \ll \alpha$. As follows from the analysis of Eqs. (A3) and (A4), unlike the previous case, for $\varepsilon \ll \alpha \ll 1$ the distribution $\theta(\xi)$ is essentially nonmonotonic both in the refractive region [at $\xi \le a$ in Fig. 1(b)] and ahead of the ASs front wall [$\xi \ge b$ in Fig. 1(b)]. These results are in agreement with the conclusions of the general theory of ASs [3,4].

III. VELOCITY AND WIDTH OF TRAVELING AUTOSOLITONS

First let us consider the case $\alpha \ll \varepsilon^4 \ll 1$, i.e., when the drift length of the inhibitor $\tilde{L}=V\tau_\eta \gg L$ but $L \gg l$. At small velocities $v \ll 1$ as follows from Eq. (A9) the width of the AS $\mathcal{L}=b-a$ is determined by the equation

$$\mathcal{L}=0.5\frac{(3+y)(1-e^{-\mathcal{L}})}{1+0.5(1+y)e^{-\mathcal{L}}}, \tag{8}$$

where $y=v^2\alpha^{-1}$. Equation (8) admits solutions if $y=v^2\alpha^{-1}\ge 1$. Therefore the minimum velocity $v_{\min} \ge \sqrt{\alpha}$. Note that this result obtained for $\alpha \ll 1$ and $L \gg l$ coincides with those obtained earlier in [6] for the case $\alpha \ll 1$ but $L \ll l$ and confirms the conclusions of the general theory of traveling ASs [3,4].

Substituting of Eq. (A3) into Eq. (A8) yields

$$A=0.5(1-e^{-\mathcal{L}})\left[1+0.5v+1.5\frac{\alpha}{v}-3\left[\frac{\alpha}{v\varepsilon}\right]^2\right]-\frac{\alpha\mathcal{L}}{v}. \tag{9}$$

Equation (9) together with Eq. (8) determines the dependencies $v(A)$ and $\mathcal{L}(A)$ shown in Figs. 2(a) and 2(b), respectively. The width and velocity of the AS decreases as A is reduced, and at a certain $A=A_{\min}\approx 0.5$ when $\mathcal{L}_{\min}\simeq 1$ and $v_{\min}\simeq\sqrt{\alpha}$ the solution in the form of a trav-

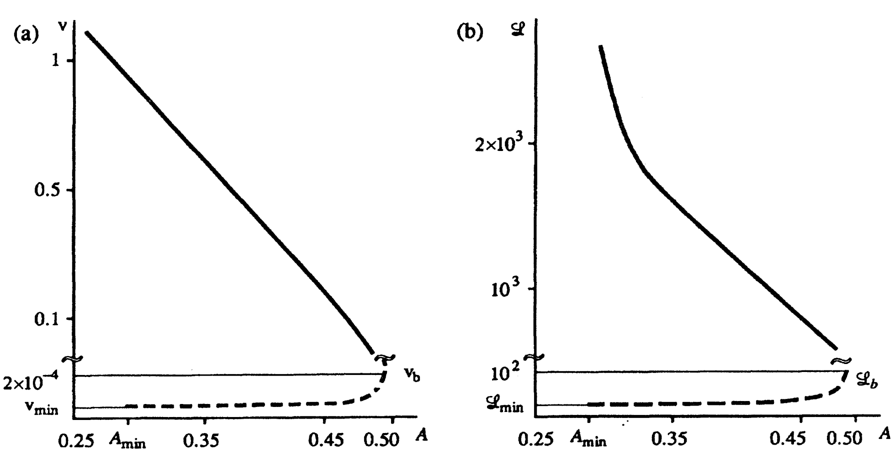

FIG. 2. Dependencies of the speed (a) and the width (b) of the traveling autosoliton on the bifurcation parameter A for the case $\alpha \ll \varepsilon^4 \ll 1$. The results of numerical calculations from Eqs. (8), (9), and (A9) for $\varepsilon=10^{-2}$ and $\alpha=10^{-9}$. $v_{\max}=2(3)^{-1/2}$, $v_b=2\times10^{-4}$, $\mathcal{L}_b=12$, $A_b=0.499$, $v_{\min}=4\times10^{-5}$, $\mathcal{L}_{\min}\simeq 1$, and $A_{\min}=0.29$.

eling AS disappears (Fig. 2). One can also find from Eqs. (8) and (9) the velocity $v=v_b$ and width $\mathcal{L}=\mathcal{L}_b$ at the point $A=A_b$ where $dv/dA=\infty$ and $d\mathcal{L}/dA=\infty$:

$$v_b=\lambda\alpha^{1/2},\quad \mathcal{L}_b=\ln(v_b\alpha^{-1})=\ln(\lambda\alpha^{-1/2}),$$

$$A_b=\frac{1}{2}\left[1-\frac{\alpha^{1/2}}{\lambda}\right]\left[1+\tfrac{1}{2}\lambda\alpha^{1/2}+\frac{3\alpha^{1/2}}{2\lambda}-3\frac{\alpha}{\lambda^2\varepsilon^2}\right]-\frac{\alpha^{1/2}}{\lambda}\ln(\lambda\alpha^{-1/2}). \tag{10}$$

It should be mentioned that the width and the velocity are measured in units of l and l/τ_θ, respectively. In Eq. (10) the coefficient λ is a numeric factor weakly dependent on the value of α, for example, $\lambda=2.2$ for $\alpha=10^{-2}$ and $\lambda=6.3$ for $\alpha=10^{-9}$. The $\mathcal{L}_b$ in turn weakly (logarithmically) depends on λ, so one can set in Eq. (10), for instance, $\lambda=3$ and assume

$$\mathcal{L}_b\simeq\ln(3\alpha^{-1/2}). \tag{11}$$

Now let us consider the case of high AS velocities. As follows from Eqs. (A5), (A6), and (A9), when $A\rightarrow 0.25$ the width of the AS $\mathcal{L}\rightarrow\infty$ and its velocity reaches a maximum,

$$v_{\max}=2(3)^{-1/2}. \tag{12}$$

Note that for $A=0.25$ the traveling AS, as well as the static one [3,4,9], takes the shape of a complex domain wall where $\theta(\xi)$ and $\eta(\xi)$ are given by Eqs. (A3)–(A6) with $(b-a)=\infty$ and $\alpha\ll\varepsilon^4\ll 1$. In other words, the value of $v_{\max}$ (12) determines the propagation velocity of such a domain wall.

Thus we have shown that in the case $\alpha\ll\varepsilon^4\ll 1$, i.e., when $L\gg l$ but $\tilde{L}=V\tau_\eta\gg L$, the velocity and width of the AS for the problem (3)–(5) lie within the ranges $\sqrt{\alpha}\lesssim v\leq 2(3)^{-1/2}$ and $1\lesssim\mathcal{L}\leq\infty$ (Fig. 2), respectively. These results practically coincide with those obtained in [3,6,7] for the case $\alpha\ll 1$ and $\varepsilon=l/L=\infty$ $(L=0)$ when the inhibitor's diffusion can be neglected.

Now let us consider the case $\varepsilon^2\ll\alpha\leq\varepsilon\ll 1$ when the inhibitor's diffusion is essential, and the function $v(A)$ depends on the ratio $\beta=\varepsilon/\alpha$ (Fig. 3) and differs qualitatively from the $v(A)$ dependence for $\alpha\ll\varepsilon^4\ll 1$ [Fig. 2(a)]. The function $v(A)$ is given by the transcendental equations (A10) and (A11) obtained from Eq. (A9) for $\varepsilon^2\ll\alpha\leq\varepsilon\ll 1$. As follows from the analysis of Eqs. (A10) and (A11), the velocity of the AS increases as its width grows. At $L\rightarrow\infty$, i.e., when $A\rightarrow 0.25$, the velocity of the AS approaches its maximum value,

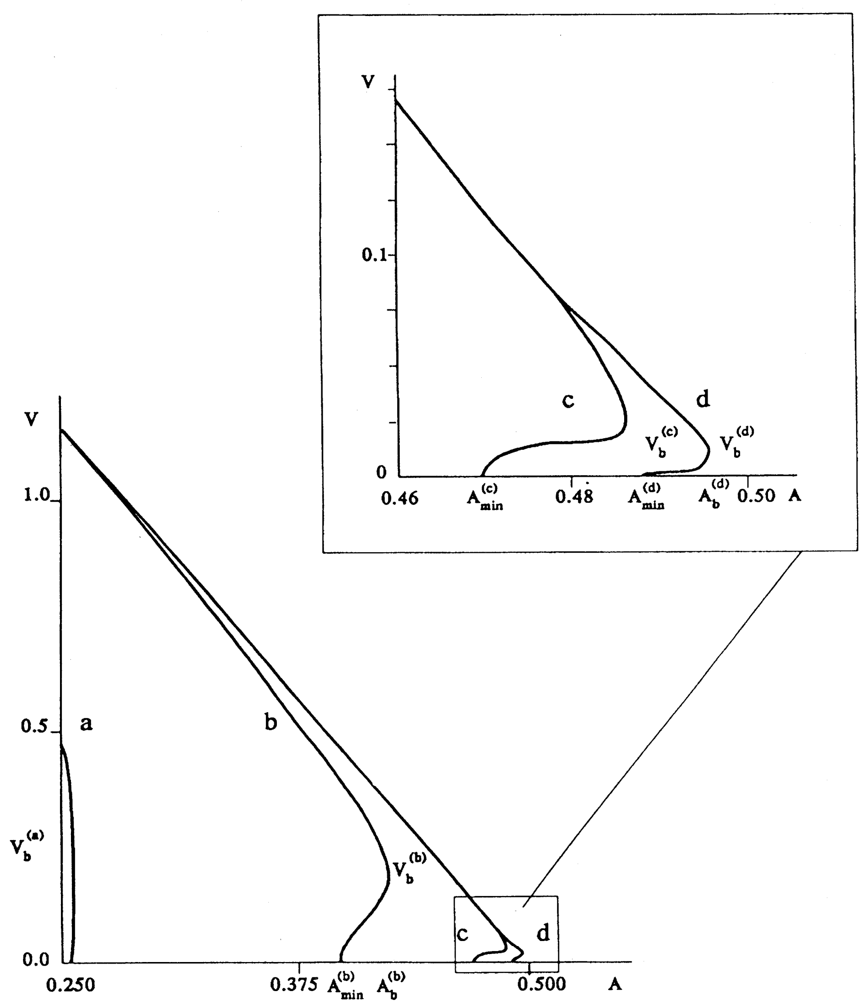

FIG. 3. Dependence of the speed of the traveling autosoliton on the bifurcation parameter A for the case $\varepsilon^4\ll\alpha\lesssim\varepsilon\ll 1$ for different values of $\beta=\varepsilon/\alpha$. Results of numerical calculations of Eqs. (A10) and (A11). Curve a: $\beta=3.1, v_{\min}=0, A_{\min}^{(a)}=0.2543$, $v_b^{(a)}=0.16$, $A_b^{(a)}=0.2545$, and $v_{\max}^{(a)}=0.47$. Curve b: $\beta=28$, $v_{\min}=0$, $A_{\min}^{(b)}=0.397$, $v_b^{(b)}=0.185$, $A_b^{(b)}=0.425$, and $v_{\max}^{(b)}=1.149$. Curve c: $\beta=360$, $v_{\min}=0$, $A_{\min}^{(c)}=0.47, v_b^{(c)}=0.02$, $A_b^{(c)}=0.485$, and $v_{\max}^{(c)}=1.1546$. Curve d: $\beta_c=2^{10.5}, v_{\min}=0$, $A_{\min}^{(d)}=0.487$, $v_b^{(d)}=0.0125$, $A_b^{(d)}=0.496$, and $v_{\max}^{(d)}=1.1547$.

$$v_{\max}=\frac{2\sqrt{\beta^2-8}}{\sqrt{3}\beta}=\frac{2}{\sqrt{3}}(1-8\alpha^2\varepsilon^{-2})^{1/2} . \tag{13}$$

Therefore, when $\varepsilon \ll 1$, the traveling AS can exist only when

$$\alpha < 2^{-3/2}\varepsilon . \tag{14}$$

Note that when $\beta=\varepsilon\alpha^{-1} \gg 8$ Eq. (13) transforms into Eq. (12) which is valid for $\alpha \ll \varepsilon^4 \ll 1$. Figures 3 and 4 show the dependencies $v(A)$ and $\mathcal{L}(A)$ obtained by computer calculations from Eqs. (A10) and (A11) for several values of the parameter β.

We would like to emphasize that, as follows from the analysis of Eq. (A10) in the considered case the velocity of the AS becomes zero at $A=A_{\min}$ (Fig. 3) while the width of the AS remains finite, equal to $\mathcal{L}_{\min}$ (Fig. 4). In other words, at the point $A=A_{\min}$ ($A_{\min}$ depends on the parameter $\beta=\varepsilon\alpha^{-1}$) the traveling AS spontaneously transforms into a static one. This means that, if the condition $\varepsilon^2 \ll \alpha \le \varepsilon \ll 1$ holds, an entirely new type of bifurcation occurs. $A=A_{\min}$ is the point where bifurcation of two qualitatively different classes of solutions—the static and automodel ones—takes place (Fig. 5).

To estimate the values of $A_{\min}$ and $\mathcal{L}_{\min}$ let us express Eqs. (A10) and (A11) as a power series in $v \ll 1$. Retaining the terms to v^2 we obtain

$$A=A_{\min}+v^2\beta^2\times 0.015[(1-2\sqrt{2}\beta^{-1})z-2\sqrt{2}\beta^{-1}(1+z)^{-2}] , \tag{15}$$

where

$$A_{\min}=0.25[1+(1-2\sqrt{2}\beta^{-1})(1+z)^{-1}] . \tag{16}$$

It also follows from Eq. (A10) that for $v \to 0$ the width of the AS is determined by the equation

$$1-2\sqrt{2}\beta^{-1}=e^{-z_m}(1+z_m),\quad z_m=\sqrt{2}\mathcal{L}_{\min}/L . \tag{17}$$

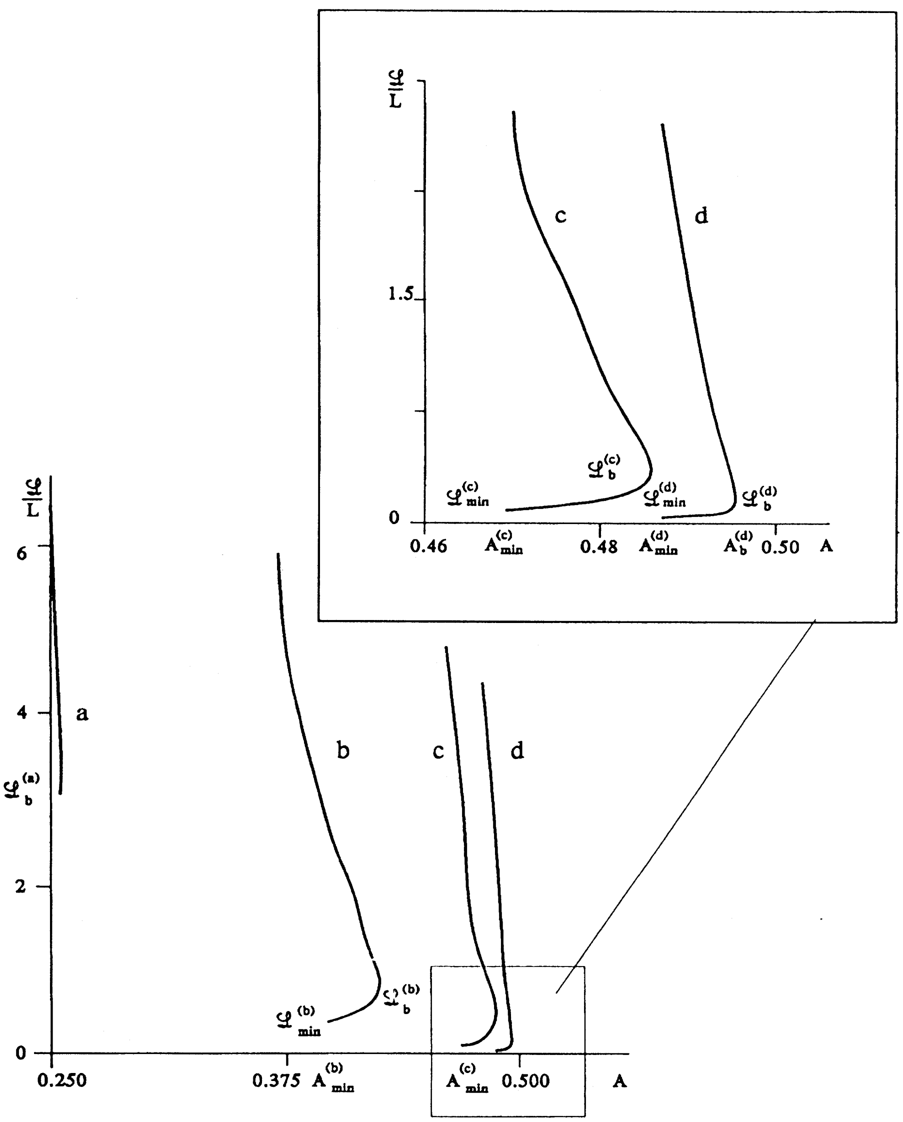

FIG. 4. Dependence of the width of the traveling autosoliton on the bifurcation parameter A for the case $\varepsilon^4 \ll \alpha \lesssim \varepsilon \ll 1$ for different values of $\beta=\varepsilon/\alpha$. Results of numerical calculations from Eqs. (A10) and (A11). Curve a: $\beta=3.1$, $\mathcal{L}_{\min}^{(a)}=3L$, $A_{\min}^{(a)}=0.2543$, $\mathcal{L}_b^{(a)}=3.1L$, and $A_b^{(a)}=0.2545$. Curve b: $\beta=28$, $\mathcal{L}_{\min}^{(a)}=0.38L$, $A_{\min}^{(b)}=0.397$, $\mathcal{L}_b^{(b)}=0.8L$, and $A_b^{(b)}=0.425$. Curve c: $\beta=360$, $\mathcal{L}_{\min}^{(c)}=0.092L$, $A_{\min}^{(c)}=0.47$, $\mathcal{L}_b^{(c)}=0.3L$, and $A_b^{(c)}=$ 0.485. Curve d: $\beta_c=2^{10.5}$, $\mathcal{L}_{\min}^{(d)}=0.045L$, $A_{\min}^{(d)}=0.487$, $\mathcal{L}_b^{(d)}=0.2L$, and $A_b^{(d)}=0.496$.

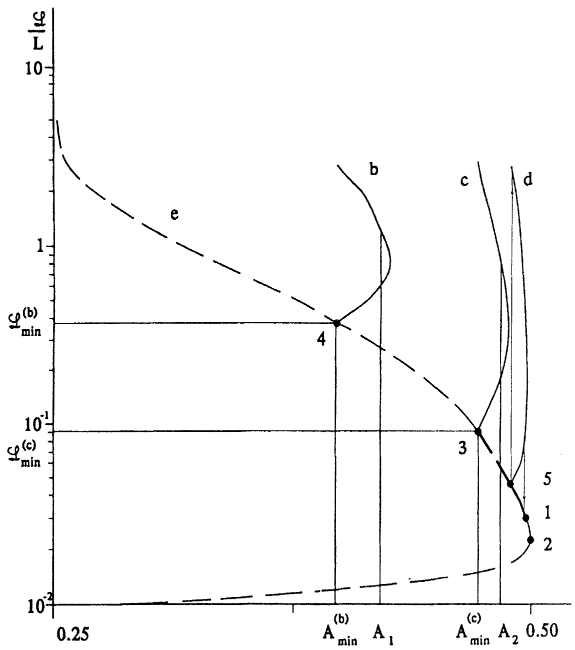

FIG. 5. Dependence of width of static (curve e) and traveling (curves b–d) autosolitons on bifurcation parameter A. Numbers 1–5 indicate various bifurcation points. 1, the bifurcation point of solutions in the form of narrow pulsating and static ASs with width $\mathcal{L}=\mathcal{L}_{b\omega}$; 2, the boundary point; 3, the bifurcation point of solutions in the form of wide pulsating and static ASs with width $\mathcal{L}=\mathcal{L}_{\omega}$; 4, the bifurcation point of unstable solutions in the form of static and traveling ASs; 5, tricritical bifurcation point of static, pulsating, and traveling ASs. Curves b–d correspond to the same curves in Fig. 4. (Dashed line indicates the portions of the curves which correspond to unstable static ASs for $\beta=\beta_c$.)

Equation (17) has solutions if the inequality (14) holds, i.e., when $\beta>1$ and $z_m<1$. Let us expand the right side of Eq. (17) in power series in z_m, retaining the terms up to z_m^2. After some algebra for $\beta>16$ we find

$$\mathcal{L}_{\min}=2^{3/4}\beta^{-1/2}L\ . \tag{18}$$

As follows from Eqs. (15)–(18) for $\beta\gg1$, in dimensional units

$$A_{\min}=0.25[1+\exp(-2^{5/4}\beta^{-1/2})]\ , \tag{19}$$

$$V=\pm\frac{10}{\sqrt{3.5}}\beta^{-3/4}\sqrt{(A-A_{\min})}\left[\frac{l}{\tau_\theta}\right]\ , \tag{20}$$

$$\mathcal{L}-\mathcal{L}_{\min}=4.5(A-A_{\min})L\ . \tag{21}$$

The formulas (20) and (21) are valid only when A is close to $A_{\min}$. Two signs in Eqs. (20) appear due to the symmetry of the problem (5)–(7) with respect to inversion of the x axis. According to Eqs. (18)–(21) $\mathcal{L}_{\min}$ decreases and $A_{\min}$ approaches 0.5 with increase of $\beta=\varepsilon/\alpha$ (Fig. 4).

IV. STABILITY OF TRAVELING AUTOSOLITON

An analytical investigation of the stability of the traveling AS is carried out in Appendix B. Its results essentially depend on the ratio between α and ε.

In the case of $\alpha\ll\varepsilon^4$ we can calculate $\Delta\Lambda_i$ and $\Delta\Omega_i$ from Eqs. (B11) and (B12) with accuracy up to the first power of $\Gamma\ll1$ and write Eq. (13) in the form

$$(y-3)(\alpha^{1/2}-y^{1/2}e^{-\mathcal{L}})=2\mathcal{L}e^{-\mathcal{L}}y^{1/2}\ , \tag{22}$$

where, as before, $y=v^2\alpha^{-1}$, and $\mathcal{L}=b-a$. Using Eq. (8) one can obtain from Eq. (22) that on the stability threshold ($A=A_c$) the critical width and velocity of a traveling AS are approximately

$$\mathcal{L}_c\simeq\ln\left[v_b\alpha^{-1}\frac{2\mathcal{L}-3}{\mathcal{L}-3}\right]=\mathcal{L}_b+\ln\left[\frac{2\mathcal{L}-3}{\mathcal{L}-3}\right]\ , \tag{23}$$

$$v_c\simeq v_b+\alpha(v_b)^{-1}\ln\left[2\frac{\mathcal{L}-2}{\mathcal{L}-3}\right]\ , \tag{24}$$

where v_b and $\mathcal{L}_b$ are the velocity and width of the AS at the point $A=A_b$ where $dv/dA=\infty$ and $d\mathcal{L}/dA=\infty$ (Fig. 2). Approximate values of $\mathcal{L}_b$ and v_b are given by Eqs. (10). According to Eqs. (10), in the case $\alpha\ll\varepsilon^4$ and $\alpha\le10^{-2}$ the $\mathcal{L}_b>3$. In view of this fact we obtain from Eqs. (23) and (24) that the stability threshold of the traveling AS lies above the points $\mathcal{L}_b$ and v_b of curves $v(A)$ and $\mathcal{L}(A)$ (Fig. 2).

Thus in the case $\alpha\ll1,\varepsilon^4$ the traveling AS loses stability with a change of bifurcation parameter A when its width $\mathcal{L}=\mathcal{L}_c$ and velocity $v=v_c$ exceed the critical values of $\mathcal{L}_b$ and v_b by values of order of themselves. Note that this result also holds for the case $L=0$. At the same time, in this case the AS's velocity has a lower limit (see Sec. III), i.e., the value $v_{\min}\neq0$ [Fig. 2(a)]. In other words, the traveling AS does not transform asymptotically into the static one at any A.

In contrast to this, in the case $\varepsilon^2\ll\alpha\le\varepsilon\ll1$ (see Sec. III) the velocity of the traveling AS becomes zero at a certain $A=A_{\min}$ (Fig. 3). In other words at $A\to A_{\min}$ the traveling AS asymptotically transforms into the static one. This means that the curve $\mathcal{L}(A)$ corresponding to the traveling AS (curves a–d in Fig. 4) links with the curve $\mathcal{L}(A)$ for the static AS at point $A=A_{\min}$ where $\mathcal{L}=\mathcal{L}_{\min}$ (in Fig. 5). The dependence $\mathcal{L}(A)$ for the static AS is determined by Eqs. (A3) and (A4), if we set $v=0$, $\xi=a$, and use that $\theta(a)=A$, $\mathcal{L}=a-b$ (see also the equations obtained in [9,3]).

It seems natural that when A is close to the point $A=A_{\min}$ the traveling AS is unstable if this point lies in the instability region of the static AS (Fig. 5). So we shall analyze the stability of the static AS in the case $\varepsilon^2\ll\alpha\le\varepsilon\ll1$ on the basis of the results presented in [3,9,10].

V. CONDITION FOR SPONTANEOUS FORMATION OF TRAVELING AND PULSATING AUTOSOLITONS

To analyze the stability of the static AS it is convenient to use the dependencies of the AS width or the value of the inhibitor in the AS wall (i.e., the value of $\eta(x)|_{x=a}=\eta_S$)) on the bifurcation parameter A [3,4]. The dependencies $\mathcal{L}(A)$ and $\eta_S(A)$ are determined by Eqs. (A3)–(A5) if we set $\xi=a$, $v=0$, and use $\theta(a)=A$, $\mathcal{L}=b-a$.

These dependencies are analyzed in [9] and their form is shown in Figs. 5 (curve e) and 6. The values of $\eta_S<0.25$ correspond to a so-called "hot" static AS [3,4]. The value of the activator in its center is greater than in its periphery [$\theta(0)>\theta_h=0$].

When $A\to 0.25$ the value of $\eta_S\to 0.25$ and $\mathcal{L}\to\infty$. At the point $A=0.25$ (point 6 in Fig. 6) the hot AS takes the shape of a complex domain wall, whereas for $A=0.25$ and $\eta_S>0.25$ it transforms into a "cold" AS. The value of the activator in the center of a cold AS is smaller than that in its periphery [$\theta(0)<\theta_h=0.5$].

Substitution of $\xi=a$ and $v=0$ into Eqs. (A4) and (A5) for $\varepsilon\ll 1$ yields the relationship between η_S and the width of the "hot" AS

$$\eta_S=0.25\left[1-\exp[-\varepsilon\sqrt{2}\mathcal{L}]\right] . \qquad (25)$$

According to the general theory of ASs [3,4] and the results of investigations of the RRRK model [9,10,3] the stability region of hot ASs with respect to aperiodic perturbations with $\mathrm{Im}\Gamma=0$ corresponds to the portion of curve $\eta_S(A)$ in Fig. 6 between points 2 and 6 (6 and 2′ for cold ASs). The lower branches of these curves in Figs. 5 and 6 (for the smaller values $\mathcal{L}$ and η_S) up to the point $A=A_b$, where $d\mathcal{L}/dA=\infty$ and $d\eta_S/dA=\infty$ (point 2 in Figs. 5 and 6), correspond to the unstable hot static AS regardless of α and ε.

The analysis of stability of the static AS for the RRRK model with respect to perturbations with $\mathrm{Im}\Gamma\equiv\omega\neq 0$,

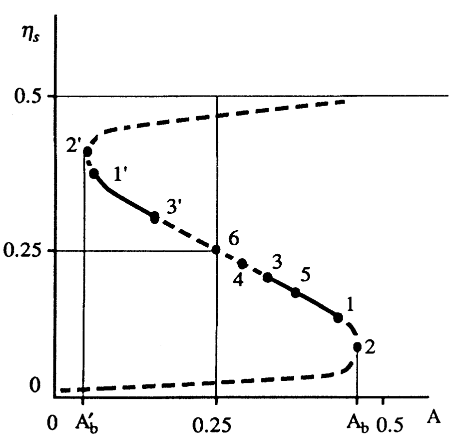

FIG. 6. Dependence of η_S (the value of the inhibitor in the wall of a static autosoliton) on bifurcation parameter A. Points 2 and 2′ are the stability boundaries with respect to aperiodic perturbations of "hot" and "cold" ASs, respectively. Point 6 corresponds to $A=0.25$ when $\mathcal{L}=\infty$. Points 1 and 3 are stability boundaries with respect to pulsations of "hot" ASs; points 1′ and 3′ are those for "cold" ASs. Bifurcation points 1–5 correspond to points 1–5 in Fig. 5. (Dashed line indicates the portions of the curves which correspond to unstable static ASs for $\beta=\beta_c$.)

i.e., pulsations, is carried out in Appendix C. The general conditions for spontaneous formation and the shape of pulsating ASs were analyzed in [3,4,13,14] and for the RRRK model in [8,3]. The properties of pulsating ASs in different models were investigated in [15–17,3,4].

Equation (C7) determines the stability threshold with respect to the growth of a critical fluctuation, describing a small change of the distance between AS walls. The growth of this perturbation (varying with the frequency $\omega_c\neq 0$) can lead to the formation of pulsating ASs [3,4]. As follows from Eq. (C7) for $\alpha<\varepsilon\ll 1$, the static AS becomes unstable ($\mathrm{Im}\omega<0$) with respect to perturbations with frequency

$$\omega=\pm\omega_c\simeq\pm 2\varepsilon^{1/2}\beta^{1/6}(\tau_\theta\tau_\eta)^{-1/2} , \qquad (26)$$

both when its width is less than

$$\mathcal{L}_{b\omega}\simeq l\ln[2(\varepsilon^2\alpha)^{-1/3}] \qquad (27)$$

and when the width is greater than

$$\mathcal{L}_\omega\simeq(\alpha/8\varepsilon)^{1/3}L=(8\beta)^{-1/3}L . \qquad (28)$$

Note that Eqs. (26)–(28) are written in dimensional units and are valid for $\varepsilon=l/L\ll 1$ and $\alpha=\tau_\theta/\tau_\eta\ll 1$. It should be emphasized that Eqs. (26)–(28), which determine the frequency of pulsations and the critical widths of static ASs, are in good agreement with the estimations for the values ω_c, $\mathcal{L}_{b\omega}$, and $\mathcal{L}_\omega$, obtained in the AS general theory [3,4]. The width $\mathcal{L}_{b\omega}$ corresponds to the point 1, and $\mathcal{L}_\omega$ corresponds to the point 3 (or 4) of the curve $\mathcal{L}(A)$ (Fig. 5, curve e). In other words, the points 1 and 3 (or 4) are the points of bifurcation of solutions in the form of one "hot" static AS and two ASs pulsating in antiphase with frequencies $\omega=\pm\omega_c$ according to Eq. (26).

Equation (25) corresponds to the minus sign in Eq. (C1), i.e., it determines the stability of the static AS with respect to the growth of a fluctuation $\delta\theta_1(x,t)$, describing a small translation of the AS along the x axis. Aperiodic growth of this fluctuation (with frequency $\omega=0$) determines spontaneous formation of the traveling AS. Equation (25) has two solutions. The first solution is trivial: $\omega=0$, i.e., $\Gamma=0$. It satisfies Eq. (25) for any value of $\mathcal{L}$ and is a consequence of translational symmetry of the problem (5)–(7) [3,4]. We can derive the nontrivial solution of Eq. (25) assuming $1\ll\mathcal{L}\ll\varepsilon^{-1}$ and expanding the expression $(2+i\omega\alpha^{-1})$ into series in $i\omega\alpha^{-1}$ where $|i\omega\alpha^{-1}|\ll 1$. As a result, retaining the terms up to the second power of $i\omega\alpha^{-1}$, we obtain

$$\Gamma=-i\omega=4\alpha B[2\alpha B^{1/2}\varepsilon^{-1}(\varepsilon\mathcal{L})^{-2}-1] . \qquad (29)$$

As follows from Eq. (29) when

$$\mathcal{L}>\mathcal{L}_{\min}=2^{3/4}\beta^{-1/2}L \qquad (30)$$

the fluctuation $\delta\theta_1$ grows periodically, leading to the spontaneous formation of a traveling AS. This is also confirmed by the fact that the condition (30), as can be seen from (18), coincides with the point $A=A_{\min}$ where the velocity of the traveling AS goes to zero, i.e., where the traveling AS transforms into the static one.

As follows from the above analysis and the form of curves $\mathcal{L}(A)$ for the traveling and static ASs (Fig. 5), the spontaneous transformation of the static AS into the traveling one and vice versa occurs abruptly and is accompanied by hysteresis.

VI. TRICRITICAL POINT AND CONDITIONS FOR COEXISTENCE OF STATIC, PULSING, AND TRAVELING AUTOSOLITONS

It was shown in Sec. III that in the case $\varepsilon^2 \ll \alpha \leq \varepsilon \ll 1$ bifurcation of solutions in the form of static and traveling ASs occurs at a certain $A = A_{\min}$ which depends on $\beta = \varepsilon/\alpha$ (Fig. 5). As follows from the results of Sec. V, at certain values of the parameter β the point $A = A_{\min}$ may lie in the instability region of the static AS with respect to the spontaneous formation of the traveling AS. At other values of β the point $A = A_{\min}$ may lie in the instability region of the static AS with respect to the spontaneous formation of the pulsating AS (point 4 in Fig. 5). Therefore, at certain $\beta = \beta_c$ the bifurcation points of solutions in the form of traveling and static ASs and also in the form of static and pulsating ASs coincide with each other.

Hence, at $\beta = \beta_c$, a tricritical point appears where the bifurcation of solutions in the form of static, pulsating, and traveling ASs takes place (point 5 in Fig. 5). At this tricritical point the minimum width of the traveling AS, $\mathcal{L}_{\min}$, must coincide with the critical width of the static AS, such that it becomes unstable with respect to pulsations.

We have shown that at the bifurcation points of static and pulsating ASs the critical width is either $\mathcal{L}_{b\omega}$ or $\mathcal{L}_{\omega}$, which are given by Eqs. (27) and (28), respectively. Note that Eqs. (27) and (28), as well as formulas (18) and (30) determining the quantity of $\mathcal{L}_{\min}$, are obtained for $\varepsilon^2 \ll \alpha \leq \varepsilon$. When this condition is fulfilled, a correct solution exists only when (18) is equal to (28). This equality is valid at $\beta = \beta_c = 2^{10.5} \simeq 1400$. At $\beta = \beta_c$ the condition $\varepsilon\beta = \varepsilon^2\alpha^{-1} \ll 1$, such that Eqs. (18) and (28) are valid, is fulfilled when $\varepsilon \ll 10^{-3}$.

Therefore, when $\varepsilon = 2^{10.5}\alpha \ll 1$ at the point $A = A_{\omega} = A_{\min}$ the bifurcation of solutions in the form of static, pulsating, and traveling ASs is realized (point 5 in Figs. 5 and 6). According to Eqs. (28) and (19) the critical width of the AS, $\mathcal{L} = \mathcal{L}_{\min}^{(c)}$, and the value $A = A_{\min}^{(c)}$ at this tricritical point, are equal to

$$\mathcal{L}_{\min}^{(c)} = 2^{-9/2}L, \quad A_{\min}^{(c)} = 0.25[1 + \exp(-\tfrac{1}{16})] \ . \tag{31}$$

VII. CONCLUSION

The main results of our analytical investigation of the RKKK model can be summarized as follows.

(1) Traveling ASs can appear when $\alpha \equiv \tau_\theta/\tau_\eta \ll 1$, where τ_θ and τ_η are the characteristic times of variation of the activator θ and inhibitor η, respectively. The velocity of traveling ASs decreases as the inhibitor's diffusion length L increases. No solutions in the form of traveling ASs exist at any A when $\varepsilon \equiv l/L < 2^{3/2}\alpha$ (l is the activator's diffusion length). These results are in agreement with the conclusions of AS general theory [3,4].

(2) When $\alpha < \varepsilon^4$, independently of ε, the solution in the form of a traveling AS disappears at the point $A = A_b$ where $dv/dA = \infty$ and $d\mathcal{L}/dA = \infty$ (Fig. 2). At this point the AS velocity is equal to the finite value $v_b \cong a^{1/2}l/\tau_\theta$. However, the traveling AS becomes unstable when its velocity is higher than v_b, i.e., at a certain $A = A_c$ less than A_b. When $A \rightarrow 0.25$ the width of the traveling AS $\mathcal{L} \rightarrow \infty$, and its velocity reaches its maximum $v_{\max} \simeq 2 \times 3^{-1/2}(l/\tau_\theta)$.

(3) When $\varepsilon^2 \ll \alpha \lesssim \varepsilon < 1$ the velocity of the AS goes to zero at a certain $A = A_{\min}$, i.e., the traveling AS transforms into the static one. On the other hand, at the point $A = A_{\min}$ the static AS becomes unstable with respect to the spontaneous formation of traveling or pulsating ASs. In other words, the bifurcation of solutions in the form of static and pulsating or traveling ASs occurs at $A = A_{\min}$.

At $\beta < \beta_c = 2^{10.5}$ the bifurcation point $A = A_{\min}$ corresponds to the instability region of static ASs with respect to pulsations (point 4 in Fig. 5). At $\beta > \beta_c$, or more exactly when $\varepsilon^2 \ll \alpha < \varepsilon\beta_c^{-1}$, this bifurcation point corresponds to the instability region of the static AS with respect to the spontaneous formation of a traveling AS. In this case solutions exist in the form of two pairs of traveling ASs in the vicinity of this point ($A = A_1$ in Fig. 5). Two ASs in the same pair differ from each other by the direction of their velocity only.

(4) At $\beta = \beta_c$, i.e., $\varepsilon = 2^{10.5}\alpha \ll 1$, a tricritical point appears (point 5 in Fig. 5), where the bifurcation of three solutions in the form of static, pulsating, and traveling ASs occurs.

(5) In certain ranges of β and A, close to β_c and $A_{\min}^{(c)}$, solutions in the form of two static, four traveling, and two pulsating ASs exist (at $A = A_2$ in Fig. 5). Only some of these solutions are stable.

When β and A are close to β_c and $A_{\min}^{(c)}$, respectively, i.e., in the vicinity of the tricritical point (5 in Fig. 5), it is possible to excite the static, pulsating, and traveling ASs simultaneously. To prove this statement it is necessary to show that the supercritical bifurcation mode of solutions describing the small amplitude AS pulsations occurs at the point $A = A_{\min}$. Such an analysis may be carried out on the basis of the multifunctional variational method of description of ASs and other dissipative-structure dynamics in active nonequilibrium systems developed in [18].

APPENDIX A

Due to piecewise linearity of the problem (3)–(5), we can use the operator method, which was applied in [9,10] for investigation of static autosolitons. After the Fourier transform of each term in Eqs. (3) and (4), we get a set of algebraic equations in terms of Fourier images. From this set we find that

$$\tilde{\theta}(\omega)=-\frac{(\omega^2+\varepsilon^2-j\omega v\alpha^{-1}\varepsilon^2)[\exp(-j\omega b)-\exp(-j\omega a)]}{j\omega\{(\omega^2+1-j\omega v)(\omega^2+\varepsilon^2-j\omega v\alpha^{-1}\varepsilon^2)+\varepsilon^3\}}\ , \tag{A1}$$

$$\tilde{\eta}(\omega)=\frac{\tilde{\theta}(\omega)\varepsilon^2}{\omega^2+\varepsilon^2-j\omega v\alpha^{-1}\varepsilon^2}\ . \tag{A2}$$

Taking the reverse Fourier transform, we obtain

$$\theta(\xi)\equiv\begin{cases}-\Lambda_2\Omega_2^{-1}e^{-\Omega_2(\xi-a)}[1-e^{\Omega_2(b-a)}]-\Lambda_4\Omega_4^{-1}e^{-\Omega_4(\xi-a)}[1-e^{\Omega_4(b-a)}], & \xi\le a,\\ \Lambda_1\Omega_1^{-1}e^{-\Omega_1(\xi-a)}+\Lambda_3\Omega_3^{-1}e^{-\Omega_3(\xi-a)}+\Lambda_2\Omega_2^{-1}e^{-\Omega_2(\xi-b)}+\Lambda_4\Omega_4^{-1}e^{-\Omega_4(\xi-b)}+0.5, & a\le\xi\le b,\\ \Lambda_1\Omega_1^{-1}e^{-\Omega_1(\xi-b)}[e^{-\Omega_1(b-a)}-1]+\Lambda_3\Omega_3^{-1}e^{-\Omega_3(\xi-b)}[e^{-\Omega_3(b-a)}-1], & \xi\ge b\ ,\end{cases} \tag{A3}$$

$$\eta(\xi)\varepsilon^{-2}=\begin{cases}-M_2e^{-\Omega_2(\xi-a)}[1-e^{\Omega_2(b-a)}]-M_4e^{-\Omega_4(\xi-a)}[1-e^{\Omega_4(b-a)}], & \xi\le a,\\ M_1e^{-\Omega_1(\xi-a)}+M_3e^{-\Omega_3(\xi-a)}+M_2e^{-\Omega_2(\xi-b)}+M_4e^{-\Omega_4(\xi-b)}+0.5\varepsilon^{-2}, & a\le\xi\le b,\\ M_1e^{-\Omega_1(\xi-b)}[e^{-\Omega_1(b-a)}-1]+M_3e^{-\Omega_3(\xi-b)}[e^{-\Omega_3(b-a)}-1], & \xi\ge b\ ,\end{cases} \tag{A4}$$

where the values Ω_i ($i=1,\ldots,4$) are the solutions of the equation

$$(1-\Omega^2+v\Omega)(\varepsilon^2-\Omega^2+v\gamma\Omega)+\varepsilon^2=0\ , \tag{A5}$$

Ω_1 and Ω_3 being the positive solutions and Ω_2 and Ω_4 being the negative solutions;

$$\Lambda_i=\frac{-\Omega_i^2+v\gamma\Omega_i+\varepsilon^2}{\prod_{j\ne i}\Omega_{ji}},\quad M_i=\frac{1}{\Omega_i\prod_{j\ne i}\Omega_{ji}}\ , \tag{A6}$$

$$\Omega_{ji}=\Omega_j-\Omega_i,\quad \gamma=\varepsilon^2\alpha^{-1}\ . \tag{A7}$$

The values of $\xi=a$ and $\xi=b$ in the solution of Eqs. (A3) and (A4) are the points where $\theta(\xi)$ equals A, i.e.,

$$\theta(a)=\theta(b)=A\ . \tag{A8}$$

Using Eq. (A3), we can rewrite Eq. (A8) as follows:

$$\sum_{i=1}^{4}(-1)^i\Lambda_i\Omega_i^{-1}[1-\exp(-|\Omega_i|\mathcal{L})]=0\ , \tag{A9}$$

where $\mathcal{L}=b-a$ is the width of the AS. Assuming $v=0$ in Eqs. (A.3)–(A.9), we obtain the solution in the form of the static AS studied in detail in [8–10] (see also [3,4]).

In the case $\varepsilon^2\ll\alpha\le\varepsilon\ll1$, using Eqs. (A5)–(A7), we can transform Eqs. (A9) into

$$\begin{aligned}\nu[(1+\nu^2z^{-2})^{-1/2}-2\sqrt{2}\beta^{-1}(1+2\nu^2\beta^{-2}z^{-2})^{-1/2}]\\ =\exp[-z(1+\nu^2z^{-2})^{1/2}][\nu\cosh(\nu)(1+\nu^2z^{-2})^{-1/2}+z\sinh(\nu)]\\ -\exp[-z\varepsilon^{-1}(\tfrac{1}{2}+\nu^2\beta^{-2}z^{-2})^{1/2}][\nu2\sqrt{2}\beta^{-1}\cosh(\nu\varepsilon^{-1}\beta^{-1})+(1+2\nu^2\beta^{-2}z^{-2})^{1/2}\sinh(\nu\varepsilon^{-1}\beta^{-1})]\\ \times(1+2\nu^2\beta^{-2}z^{-2})^{-1/2}\ ,\end{aligned} \tag{A10}$$

where $z=\sqrt{2}(\mathcal{L}/L)$, $\nu=v\beta(\mathcal{L}/2L)$, and $\beta=\varepsilon\alpha^{-1}$

Equation (A8), or more precisely the equation $\theta(a)=A$, can be rewritten

$$\begin{aligned}A=&\frac{1}{2}\left[1+\frac{v^2}{4}\right]^{-1/2}\left[\left[1+\frac{v^2}{4}\right]^{1/2}-\frac{v}{2}\right]^{-1}\\ &-\frac{1}{4}\left[1+\frac{\nu}{z}\left[1+\frac{\nu^2}{z^2}\right]^{-1/2}\right]\\ &\times\left\{1-\exp\left[\nu-z\left[1+\frac{\nu^2}{z^2}\right]^{1/2}\right]\right\}\ .\end{aligned} \tag{A11}$$

Equations (A10) and (A11) determine the dependencies of width and velocity of the traveling AS on the model's parameters.

APPENDIX B

To investigate the stability of the traveling AS in the one-dimensional case, let us linearize Eqs. (1) and (2) and boundary conditions (7) with respect to fluctuations of the following type:

$$\delta\theta(\xi,t)=\delta\theta(\xi)\exp\left[-\frac{\Gamma t}{\tau_\theta}\right],$$

$$\delta\eta(\xi,t)=\delta\eta(\xi)\exp\left[-\frac{\Gamma t}{\tau_\theta}\right], \tag{B1}$$

in the vicinity of the automodel solution $\theta(\xi),\eta(\xi)$ of these equations. We obtain

$$v\frac{d}{d\xi}\delta\theta+\frac{d^2}{d\xi^2}\delta\theta=(1-\Gamma)\delta\theta+\delta\eta-\frac{\delta\theta}{\delta(\theta-A)}\ , \qquad (B2)$$

$$\alpha^{-1}v\frac{d}{d\xi}\delta\eta+\varepsilon^{-2}\frac{d^2}{d\xi^2}\delta\eta=(1-\alpha^{-1}\Gamma)\delta\eta-\delta\theta\ , \qquad (B3)$$

$$\delta\theta(\xi)=0,\ \delta\eta(\xi)=0\ \text{ for } \xi\to\pm\infty\ . \qquad (B4)$$

Here $\delta(\theta-A)$ is Dirac's delta function,

$$v=V\frac{\tau_\theta}{l},\ \ \xi=\frac{x}{l}-v\frac{t}{\tau_\theta}\ . \qquad (B5)$$

It follows from an analysis of Eqs. (B2)–(B4) similar to that in [10] for a static AS that critical fluctuations of the activator $\delta\theta(\xi)$ are localized in the walls of the traveling AS, i.e., in the vicinity of the points $\zeta=a$ and $\zeta=b$ in Fig. 1. The most dangerous fluctuations $\delta\theta_0$ and $\delta\theta_1$ describe a small change of the distance between the AS walls or its small translation along the x axis. Solutions of Eqs. (B3) and (B4) are rather unwieldy. For this reason we will only analyze the equations which link the quantities

$$\delta\theta|_{\xi=a}=\delta\theta_a \ \text{ and } \ \delta\theta|_{\xi=b}=\delta\theta_b\ , \qquad (B6)$$

i.e., the values of the activator's fluctuations in the AS wall. Carrying out transformations similar to those used for the derivation of Eqs. (A3) and (A4) we obtain

$$\delta\theta_a(\Delta\Lambda_2+\Delta\Lambda_4+\Lambda_{20}e^{\Omega_{20}\mathcal{L}}+\Lambda_{40}e^{\Omega_{40}\mathcal{L}})$$
$$=-\delta\theta_b\frac{\theta_a'}{|\theta_b'|}(\Lambda_2 l^{\Omega_2\mathcal{L}}+\Lambda_4 l^{\Omega_4\mathcal{L}})\ , \qquad (B7)$$

$$\delta\theta_b(\Delta\Lambda_1+\Delta\Lambda_3+\Lambda_{10}e^{-\Omega_{10}\mathcal{L}}+\Lambda_{30}e^{-\Omega_{30}\mathcal{L}})$$
$$=-\delta\theta_a\frac{|\theta_b'|}{\theta_a'}(\Lambda_1 e^{-\Omega_1\mathcal{L}}+\Lambda_3 e^{-\Omega_3\mathcal{L}})\ . \qquad (B8)$$

Here

$$\theta_a'=\frac{\partial\theta}{\partial\xi}\bigg|_{\xi=a}=\Lambda_{20}(1-e^{\Omega_{20}\mathcal{L}})+\Lambda_{40}(1-e^{\Omega_{40}\mathcal{L}})\ , \qquad (B9)$$

$$|\theta_b'|=\left|\frac{\partial\theta}{\partial\xi}\bigg|_{\xi=b}\right|=\Lambda_{20}+\Lambda_{40}+\Lambda_{10}e^{-\Omega_{10}\mathcal{L}}+\Lambda_{30}e^{-\Omega_{30}\mathcal{L}}\ , \qquad (B10)$$

$$\Lambda_{i\Gamma}=\frac{-\Omega_i^2+\Omega_i v\gamma-\Gamma\gamma+\varepsilon^2}{\prod_{j\neq i}\Omega_{ji}}\ , \qquad (B11)$$

where $\gamma=\varepsilon^2\alpha^{-1}$, $\Delta\Lambda_i=\Lambda_{i\Gamma}-\Lambda_{i0}$, $\Delta\Omega_i=\Omega_{i\Gamma}-\Omega_{i0}$, $\Omega_{ij}=\Omega_i-\Omega_j$, and $\Omega_{i\Gamma}$ are the solutions of the equation

$$\Omega_{i\Gamma}^4-\Omega_{i\Gamma}^3 v(1+\gamma)-\Omega_{i\Gamma}^2(1-\Gamma(1-\gamma)+\varepsilon^2-\gamma v^2)$$
$$+\Omega_{i\Gamma}(v\gamma(1-2\Gamma)+\varepsilon^2 v)+\varepsilon^2+(1-\Gamma)(\varepsilon^2-\gamma\Gamma)=0\ . \qquad (B12)$$

One can see from (B11), (B12), (A5), and (A6) that for $\Gamma=0$ the values are $\Lambda_{i0}=\Lambda_i$, and $\Omega_{i0}=\Omega_i$, i.e., they are the solutions of Eq. (A5).

Let us analyze the case of an unstable traveling AS with respect to a periodic perturbation with $\mathrm{Im}\Gamma=0$. The case $\mathrm{Im}\Gamma\neq0$, i.e., the instability of the traveling AS with respect to periodic perturbations (pulsations), will be discussed below.

When $\mathrm{Im}\Gamma=0$ on the instability boundary where $\mathrm{Re}\Gamma=0$, i.e., in the vicinity of the point $\Gamma=0$, in calculating the values $\Delta\Lambda_i$ and $\Delta\Omega_i$ we may retain only the terms up to the first power of Γ. The set of Eqs. (B7) and (B8) has a nontrivial solution for $\Gamma=0$ when

$$(\Lambda_{20}e^{\Omega_{20}\mathcal{L}}+\Lambda_{40}e^{\Omega_{40}\mathcal{L}})\{\Delta\Lambda_2(1-e^{\Omega_{20}\mathcal{L}})+\Delta\Lambda_4(1-e^{\Omega_{40}\mathcal{L}})+\Delta\Lambda_1(1-e^{-\Omega_{10}\mathcal{L}})+\Delta\Lambda_3(1-e^{-\Omega_{30}\mathcal{L}})$$
$$-\mathcal{L}[\Lambda_{20}e^{\Omega_{20}\mathcal{L}}\Delta\Omega_2+\Lambda_{40}e^{\Omega_{40}\mathcal{L}}\Delta\Omega_4-\Lambda_{10}e^{-\Omega_{10}\mathcal{L}}\Delta\Omega_1-\Lambda_{30}e^{-\Omega_{30}\mathcal{L}}\Delta\Omega_3]\}=0\ . \qquad (B13)$$

According to (A8), the condition $\theta(a)=\theta(b)=A$ is satisfied at the points $\xi=a$ and $\xi=b=a+\mathcal{L}$ regardless of the values of $\mathcal{L}$ and A. One can also see from Eq. (A3) that $\theta(a)$ and $\theta(b)$ do not explicitly depend on the values of A,a, and b, but only on $\mathcal{L}=b-a$. Therefore

$$\frac{\partial\theta(a)}{\partial\mathcal{L}}\left[\frac{\partial\mathcal{L}}{\partial A}\right]=\frac{\partial\theta(b)}{\partial\mathcal{L}}\left[\frac{\partial\mathcal{L}}{\partial A}\right]=1\ , \qquad (B14)$$

i.e.,

$$\frac{\partial\theta(a)}{\partial\mathcal{L}}=\frac{\partial\theta(b)}{\partial\mathcal{L}}\ . \qquad (B15)$$

It is easy to obtain from Eq. (A3) that

$$\frac{\partial\theta(a)}{\partial\mathcal{L}}=\Lambda_{20}e^{\Omega_{20}\mathcal{L}}+\Lambda_{40}e^{\Omega_{40}\mathcal{L}}\ ,$$
$$\frac{\partial\theta(b)}{\partial\mathcal{L}}=\Lambda_{10}e^{-\Omega_{10}\mathcal{L}}+\Lambda_{30}e^{-\Omega_{30}\mathcal{L}}\ , \qquad (B16)$$

i.e., according to Eq. (B15),

$$\Lambda_{20}e^{\Omega_{20}\mathcal{L}}+\Lambda_{40}e^{\Omega_{40}\mathcal{L}}=\Lambda_{10}e^{-\Omega_{10}\mathcal{L}}+\Lambda_{30}e^{-\Omega_{30}\mathcal{L}}\ . \qquad (B17)$$

Equation (B17) was used to derive Eq. (B13). At the point where $\partial\theta(a)/\partial\mathcal{L}=0$, Eq. (B13) is satisfied. So, when the second factor in the left-hand side of Eq. (B13) does not become zero for large values of $\mathcal{L}$, the stability boundary is determined by the condition $\partial\theta(a)/\partial\mathcal{L}=0$. Taking into account that according to Eq. (A8) the derivative $\partial\theta(a)/\partial A=1$, the condition $\partial\theta(a)/\partial\mathcal{L}=0$ can be rewritten as $\partial A/\partial\mathcal{L}=0$ or $\partial\mathcal{L}/\partial A=\infty$. This situation is realized in the case of a static AS when $a>1$, for which the stability boundary is the point where $\partial\mathcal{L}/\partial A=\infty$ [3,4,9,10]. Another situation is realized when $\alpha\ll 1$ both for static and traveling ASs.

APPENDIX C

An equation determining the stability of the static AS can be derived from the condition of existence of nontrivial solutions of Eqs. (B7) and (B8) if we set $v=0$ and $\Gamma=-i\omega$ in Eqs. (B7)–(B12). This condition can be written as

$$\Delta\Lambda_2+\Delta\Lambda_4+\Lambda_{20}e^{\Omega_{20}\mathcal{L}}+\Lambda_{40}e^{\Omega_{40}\mathcal{L}} \pm[\Lambda_{2\Gamma}e^{\Omega_2\mathcal{L}}+\Lambda_{4\Gamma}e^{\Omega_4\mathcal{L}}]=0\ . \quad \text{(C1)}$$

When Eq. (C1) is satisfied the considered equations have two solutions: one symmetrical with respect to the center of the AS, such that $\delta\theta_a=\delta\theta_b$, and one antisymmetrical, such that $\delta\theta_a=-\delta\theta_b$. The plus sign in Eq. (C1) corresponds to the symmetrical fluctuation and the minus sign to the antisymmetrical one. The first describes a small change of the distance between the ASs walls and the second describes a small translation of the AS along the x axis [3,9,10]. In Eq. (C1), as in Appendix B, $\Delta\Lambda_i=\Lambda_{i\Gamma}-\Lambda_{i0}$, where the subscript 0 signifies that the corresponding values are taken at $\omega=0$; $\Omega_{3,2},\Omega_{1,4}$ are the solutions of Eq. (B12). At $v=0$, $\Gamma=-i\omega$, and $\varepsilon\ll 1$ they are

$$\Omega_{3,2}=\pm\sqrt{2}\varepsilon[1+i\omega(2\alpha)^{-1}]^{1/2}\ , \quad \text{(C2)}$$

$$\Omega_{1,4}=\pm[1+i\omega]^{1/2}\ . \quad \text{(C3)}$$

Under the same conditions, according to Eq. (B11) the coefficients $\Lambda_{i\Gamma}$ are equal to

$$\Lambda_{3\Gamma}=-\Lambda_{2\Gamma}=0.5\varepsilon\{2[1+i\omega(2\alpha)^{-1}]\}^{-1/2}\ , \quad \text{(C4)}$$

$$\Lambda_{1\Gamma}=-\Lambda_{4\Gamma}=0.5(1-0.5i\omega)\ . \quad \text{(C5)}$$

Substituting Eqs. (C2)–(C5) in Eq. (C1), we obtain

$$-i\omega=\frac{2\varepsilon}{\sqrt{2}}[1-\exp(-\sqrt{2}\varepsilon\mathcal{L})]-\{2e^{-\mathcal{L}}\pm(2-i\omega)e^{-(1+i\omega)^{1/2}\mathcal{L}}\}-\frac{2\varepsilon}{(2+i\omega\alpha^{-1})^{1/2}}\left\{1\pm\exp[-\varepsilon\mathcal{L}(2+i\omega\alpha^{-1})^{1/2}]\right\}\ . \quad \text{(C6)}$$

Note that for $\mathcal{L}\gg l$ Eq. (C6) obtained for the case $\varepsilon\ll 1$ coincides, except for designations, with Eq. (5.4) of [12], where one should put $k=0$ and $B=1$. It confirms the correctness of the approach developed in [12] for $L\gg l$ and $\mathcal{L}\gg l$, based on the derivation of the equations describing dynamics of AS walls.

According to the AS general theory [3,4], for $\varepsilon\ll 1$ and $\alpha\ll 1$ the static AS loses stability when $\mathcal{L}\ll L$ ($\varepsilon\mathcal{L}\ll 1$). Expanding exponents in Eq. (C6) into series with respect to $\varepsilon\mathcal{L}$ and retaining only the first nonzero terms, we obtain

$$-i\omega=4\varepsilon^2\mathcal{L}-4\varepsilon(2+i\omega\alpha^{-1})^{-1/2}-2e^{-\mathcal{L}} -(2-i\omega)e^{-(1+i\omega)^{1/2}\mathcal{L}}\ , \quad \text{(C7)}$$

$$-i\omega=\varepsilon^3\mathcal{L}^2[(2+i\omega\alpha^{-1})^{1/2}-\sqrt{2}]-2l^{-\mathcal{L}} +(2-i\omega)l^{-(1+i\omega)^{1/2}\mathcal{L}}\ . \quad \text{(C8)}$$

Equations (C7) and (C8) correspond to Eq. (C1) with the plus and minus signs, and respectively, i.e., they determine the stability thresholds of static ASs with respect to the growth of symmetrical and antisymmetrical critical fluctuations.

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